

# ON THE COMPLEX ŁOJASIEWICZ INEQUALITY WITH PARAMETER

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ABSTRACT. We prove a continuity property in the sense of currents of a continuous family of holomorphic functions which allows us to obtain a Łojasiewicz inequality with an effective exponent independent of the parameter.

## 1. INTRODUCTION

The *Łojasiewicz inequality* introduced in [L] is one of the most important tools in singularity theory, both complex and real. The first result concerning a parametrized family — but, of course, with *an exponent that is independent of the parameter* — is due to Łojasiewicz and Wachta [LW]. Fairly recently, we have obtained in [D4] an effective Łojasiewicz inequality with parameter in complex analytic geometry, using only complex analytic methods. This article is somehow a continuation of that work, inspired to some extent by the observations made in [D3] and the intersection theory results introduced in [T].

Our best results are presented in the following theorem. Throughout the paper we assume that the topological space  $T$  is *1st countable*.

**Theorem 1.1.** *Assume that  $f: T \times \Omega \rightarrow \mathbb{C}$  is a continuous function where  $T$  is a locally compact, connected topological space,  $\Omega \subset \mathbb{C}^m$  is a domain, and for all  $t \in T$ ,  $f_t \in \mathcal{O}(\Omega)$  does not vanish identically. Assume moreover that  $0 \in \Omega$  and  $f_t(0) = 0$  for any  $t$ . Then*

- (1)  $Z_{f_t} \rightarrow Z_{f_0}$  in the sense of currents, where  $Z_{f_t}$  denotes the cycle of zeroes of  $f_t$ ;
- (2) there is a neighbourhood  $U \subset \Omega$  of zero in which, for all  $t$  close enough to  $t_0$ ,

$$|f_t(x)| \geq c(t) \text{dist}(x, f_t^{-1}(0))^\alpha,$$

where  $c(t) > 0$  is a constant depending on the parameter, but the exponent  $\alpha = \text{ord}_0 f_0$  is uniform.

For the convenience of the reader let us recall two basic notions of convergence of sets, especially useful in analytic geometry (see e.g. [DD] and

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[TW1]). We consider the following situation:  $T$  is a topological space and  $E \subset T \times \mathbb{R}^n$  is a set with closed sections  $E_t = \{x \in \mathbb{R}^n \mid (t, x) \in E\}$  and we put  $F := \pi(E)$  for  $\pi(t, x) = t$ . Assume that  $t_0$  is an accumulation point of  $F$ .

**Definition 1.2.** (see e.g. [DD]) We say that  $E_t$  *converges in the sense of Kuratowski* to a set  $A$ , when  $t \rightarrow t_0$ , if

- for any  $x \in A$ , for any neighbourhood  $U$  of  $x$ , there is a neighbourhood  $V$  of  $t_0$  such that  $U \cap E_t \neq \emptyset$  for all  $t \in V \cap F \setminus \{t_0\}$ ;
- if  $x$  is such that for any neighbourhood  $U \ni x$  and any neighbourhood  $V \ni t_0$  there is a point  $t \in V \setminus \{t_0\}$  such that  $U \cap E_t \neq \emptyset$ , then  $x \in A$ .

We write then  $E_t \xrightarrow{K} A$ .

If for each  $t_0$ ,  $E_t \xrightarrow{K} E_{t_0}$ , then we say that  $E$  *has continuously varying fibres*.

*Remark 1.3.* It is easy to see (cf. [TW1], [DD]) that this convergence for the graphs of a sequence continuous functions is precisely the *local uniform convergence* of the functions themselves.

We have the following straightforward observation:

**Lemma 1.4.** *If any point in  $T$  has a countable basis of neighbourhoods, then  $E_t \xrightarrow{K} A$  when  $t \rightarrow t_0$  iff*

- if  $x \in A$ , then for any sequence  $t_\nu \rightarrow t_0$  we can find points  $E_{t_\nu} \ni x_\nu \rightarrow x$ ;
- if  $x$  is such that there is a sequence  $t_\nu \rightarrow t_0$  and points  $E_{t_\nu} \ni x_\nu \rightarrow x$ , then  $x \in A$ .

In complex analytic geometry this kind of convergence is very useful for different purposes (Bishop's Theorem, algebraic approximation as in [B] or algebraicity criteria as in [DP]). We may refine it taking into account multiplicities (cf. [T] and [Ch]). In order to do so, consider a sequence of *positive pure  $k$ -dimensional analytic cycles* <sup>(1)</sup>  $Z_\nu$ ,  $\nu = 0, 1, 2, \dots$  in some  $\Omega \subset \mathbb{C}^m$  (of course, everything can be carried over to manifolds).

**Definition 1.5** (Tworzewski [T]). We say that  $Z_\nu$  *converges to  $Z_0$  in the sense of Tworzewski*, if

- the supports  $|Z_\nu| \xrightarrow{K} |Z_0|$ ;
- for any regular point  $a \in \text{Reg}|Z_0|$  and any relatively compact manifold  $M$  of complementary dimension, transversal to  $|Z_0|$  and  $a$  and

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<sup>1</sup>A positive pure  $k$ -dimensional cycle  $Z$  is a formal sum  $\sum \alpha_i S_i$  where  $\alpha_i > 0$  are integers and  $\{S_i\}$  is a locally finite family of irreducible  $k$ -dimensional analytic sets; then the analytic set  $|Z| := \bigcup S_i$  is called the *support* of  $Z$ ; for details see [T].

such that  $\overline{M} \cap |Z_0| = \{a\}$ , we have for the *total number of intersection* <sup>(2)</sup>  $\deg(Z_\nu \cdot M) = \deg(Z_0 \cdot M)$  from some index  $\nu_0$  onwards.

We will call  $M$  a *testing manifold* for  $Z_0$  at  $a$ .

*Remark 1.6.* As noted by Alain Yger [Y], this convergence is precisely *the weak convergence of the corresponding integration currents*  $[Z_\nu]$ . See also the general though not very precise discussion in [Ch].

By [T] Lemma 3.2 it is sufficient to consider testing manifolds at a dense subset of the regular points of  $|Z_0|$ .

Of course, the definition may be extended to families  $\{Z_t\}$  where  $t$  belongs to a topological space  $T$ .

It will be useful to state clearly the following observation being a mere corollary to the result of [TW1]:

**Proposition 1.7.** *If  $X_0, Y_0$  are analytic subsets of an open set  $\Omega \subset \mathbb{C}^m$  of pure dimensions  $p, q$  respectively, and if  $X_0 \cap Y_0$  has pure dimension  $p+q-m$ , then for any sequences  $X_\nu \xrightarrow{K} X_0$  and  $Y_\nu \xrightarrow{K} Y_0$  of analytic subsets of  $\Omega$  of pure dimension  $p$  and  $q$  respectively, the intersections  $X_\nu \cap Y_\nu$  are proper (i.e. of pure dimension  $p+q-m$ ) for all indices large enough.*

*Proof.* By [TW1] we know that  $X_\nu \cap Y_\nu \xrightarrow{K} X_0 \cap Y_0$ . Besides, at any  $a \in X_\nu \cap Y_\nu$  we obviously have  $\dim_a X_\nu \cap Y_\nu \geq p+q-m$ .

Now fix a point  $a \in X_0 \cap Y_0$  and choose coordinates in such a way that in a bounded neighbourhood  $W = U \times V \subset \mathbb{C}^{p+q-m} \times \mathbb{C}^{2m-p-q}$  of  $a$  the natural projection onto  $U$  restricted to the set  $Z_0 = X_0 \cap Y_0$  is a branched covering. We may ask that  $(\overline{U} \times \partial V) \cap Z_0 = \emptyset$ . Write  $Z_\nu := X_\nu \cap Y_\nu \cap W$ . Then, by the convergence, for all indices large enough,  $(\overline{U} \times \partial V) \cap Z_\nu = \emptyset$ , whereas  $Z_\nu \neq \emptyset$ .

This means that any such  $Z_\nu$  projects properly on  $U$ . Therefore, if we pick a point  $z \in Z_\nu$  and an arbitrarily small polydisc around it, then by the Remmert Proper Map Theorem,  $\dim_z Z_\nu \leq p+q-m$ . This implies that all the  $Z_\nu$ 's have pure dimension  $p+q-m$ .

Since any subsequence of  $X_\nu \cap Y_\nu$  converges to  $X_0 \cap Y_0$  the proof is accomplished.  $\square$

Finally, we briefly recall the notion of *c-holomorphic functions* (cf. [R] and [Wh]) i.e. complex continuous functions that are defined on an analytic set  $A$  and holomorphic at its regular points  $\text{Reg } A$ . We denote by  $\mathcal{O}_c(A)$  their ring for a fixed  $A$ . Their study from the geometric point of view was carried to some extent in [D1]–[D4]. They share many a property of holomorphic functions, though they form a larger class without really useful differential properties. Their main feature is the fact that they are characterized among all the continuous functions  $A \rightarrow \mathbb{C}$  by the analyticity of their graphs (see

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<sup>2</sup>By [TW1], almost all intersections  $|Z_\nu| \cap M$  are discrete and so finite. Then the total number of intersection is the formal sum of the intersection points with their respective Draper intersection indices [Dr] taken into account.

[Wh]). That allows the use of geometric methods. In particular there is an identity principle on irreducible sets (cf. [D2]) and we can consider the *order of vanishing* (see [D1] where it is introduced and studied) at a point  $f(a) = 0$  (when  $f \not\equiv 0$ ) as

$$\text{ord}_a f := \max\{\eta > 0 \mid |f(x)| \leq \text{const.} \|x\|^\eta, \text{ in a neighbourhood of } a \in A\}.$$

## 2. CONTINUITY PRINCIPLE

**Lemma 2.1.** *Let  $E \subset \mathbb{R}_t^k \times \mathbb{R}_x^n$  be a closed, nonempty set with continuously varying sections  $E_t$  over  $F := \pi(E)$  where  $\pi(t, x) = t$ . Then the function*

$$\delta(t, x) := \text{dist}(x, E_t), \quad (t, x) \in F \times \mathbb{R}^n$$

*is continuous.*

*Proof.* The function  $\delta(t, \cdot)$  is 1-Lipschitz which means that  $\lim_{x \rightarrow x_0} \delta(t, x) = \delta(t, x_0)$  is uniform with respect to  $t$ . Therefore, in view of the Iterated Limits Theorem, we need only to check that  $t \mapsto \delta(t, x)$  is continuous for all  $x$ . Indeed, then

$$\lim_{(t,x) \rightarrow (t_0,x_0)} \delta(t, x) = \lim_{x \rightarrow x_0} \delta(t_0, x) = \delta(t_0, x_0).$$

Fix  $(t_0, x_0)$ . We know that  $E_t \rightarrow E_{t_0}$  in the sense of Kuratowski. Then let  $d := d(x_0, E_{t_0})$ . In particular, for any  $\varepsilon > 0$ ,

$$(K) \quad \mathbb{B}(x_0, d + \varepsilon) \cap E_{t_0} \neq \emptyset \text{ and } \overline{\mathbb{B}}(x_0, d - \varepsilon) \cap E_{t_0} = \emptyset.$$

Then, the convergence implies (cf. [DD] Lemma 2.1) that for all  $t$  sufficiently close to  $t_0$ , condition (K) holds for  $E_t$  instead of  $E_{t_0}$ . That in turn implies that for all such  $t$ ,

$$d - \varepsilon < \text{dist}(x_0, E_t) < d + \varepsilon$$

and the proof is complete.  $\square$

*Remark 2.2.* Of course, the lemma is true for a product of metric spaces. In particular we can replace the parameter space  $\mathbb{R}^k$  by a 1st countable topological space  $T$ , since for such a  $T$  the following general Iterated Limits Theorem holds <sup>(3)</sup>: if  $f: T \times X \rightarrow Y$  where  $X, Y$  are metric spaces with  $Y$  complete, is such that

- $\exists \lim_{t \rightarrow t_0} f(t, x) = \varphi(x)$  for any  $x \in X$ ;
- $\exists \lim_{x \rightarrow x_0} f(t, x) = \psi(t)$  uniformly in  $t$ ,

then there exists  $\lim_{(t,x) \rightarrow (t_0,x_0)} f(t, x) = \lim_{x \rightarrow x_0} f(t_0, x) = \psi(t_0)$ .

**Proposition 2.3.** *Consider a pure  $(n + k)$ -dimensional analytic set  $A \subset \mathbb{C}^p \times D$  with proper projection  $\pi(t, z, w) = (t, z)$  onto the product domain  $D = U \times V \subset \mathbb{C}^n \times \mathbb{C}^k$ . Then*

- (1) *The sections  $A_t$  vary continuously;*
- (2) *The function  $\delta: \mathbb{C}^p \times D \ni (t, x) \mapsto \text{dist}(x, A_t) \in \mathbb{R}$  is continuous.*

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<sup>3</sup>We do not have a reference for this fact, but the proof is obvious.

*Proof.* Since  $A$  is closed, the sections  $A_t$  are upper semi-continuous, by [DD] Proposition 2.7, i.e. for any  $t_0$ ,

$$\limsup_{t \rightarrow t_0} A_t \subset A_{t_0}.$$

We need to check that  $A_{t_0} \subset \liminf_{t \rightarrow t_0} A_t$ . This amounts to proving that for any  $x \in A_{t_0}$  and any  $t_\nu \rightarrow t_0$  we can find points  $x_\nu \in A_{t_\nu}$  converging to  $x$ . Since  $\pi$  is a branched covering on  $A$ , we see that the fibres  $\pi^{-1}(\pi(t_\nu, x)) \cap A$  converge to the fibre  $\pi^{-1}(\pi(t_0, x)) \cap A$  containing  $(t_0, x)$  which gives exactly what we need and the proof of (1) is complete.

Now (2) follows from the previous lemma.  $\square$

*Remark 2.4.* We stress once again that (2) is a simple consequence of (1).

**Lemma 2.5.** *Let  $T$  be a locally compact topological space and  $X \subset \mathbb{C}^m$  a nonempty set. If  $f: T \times X \rightarrow \mathbb{C}$  is continuous and we write  $f_t(x) = f(t, x)$ , then  $t \rightarrow t_0$  in  $T$  implies the convergence of graphs:*

$$\Gamma_{f_t} \xrightarrow{K} \Gamma_{f_{t_0}}.$$

*Proof.* Note that the graphs in question are pure  $k$ -dimensional sets (cf. [D1]). In view of Remark 1.3 we need only to check that for any  $t_\nu \rightarrow t_0$ ,  $f_{t_\nu} \rightarrow f_{t_0}$  locally uniformly on  $X$ . Take a compact set  $K \subset X$ . Then  $K' = \{t_0\} \times K$  is compact and for a fixed  $\varepsilon$  and any  $x \in K$  we find neighbourhoods  $U_x \times \mathbb{B}(x, r_x)$  of  $(t_0, x)$  at points  $(t, y)$  of which

$$|f(t, y) - f(t_0, x)| < \varepsilon.$$

By compacity we choose a finite covering  $K' \subset \bigcup_{i=1}^p U_i \times \mathbb{B}(x_i, r_i)$  and put  $U := \bigcap_{i=1}^p U_i$ . then for any  $(t, x) \in U \times K$  we have  $(t, x) \in U_i \times \mathbb{B}(x_i, r_i)$  for some  $i$  and so

$$|f(t, x) - f(t_0, x)| \leq \varepsilon.$$

This ends the proof.  $\square$

**Proposition 2.6.** *Let  $T$  be a locally compact, connected topological space,  $A$  a pure  $k$ -dimensional analytic subset of some open set  $\Omega \subset \mathbb{C}^m$  and  $f: T \times A \rightarrow \mathbb{C}$  a continuous function such that for each  $t \in T$ ,  $f_t(x) := f(x, t)$  is  $c$ -holomorphic on  $A$ . Then  $t \rightarrow t_0$  in  $T$  implies*

$$\Gamma_{f_t} \xrightarrow{T} \Gamma_{f_{t_0}}.$$

*Proof.* By Lemma 2.5 we have

$$\Gamma_{f_t} \xrightarrow{K} \Gamma_{f_{t_0}}.$$

This means that on  $\text{Reg}A$ , for any  $t_\nu \rightarrow t_0$ , we have a sequence of holomorphic functions converging locally uniformly.

Now, observe that for any  $g \in \mathcal{O}_c(A)$ ,  $\Gamma_{g|_{\text{Reg}A}} \subset \text{Reg}\Gamma_g$  is dense. For a testing  $M$  at  $a \in \Gamma_{f_{t_0}|_{\text{Reg}A}}$  we have the equality  $M \cap T_a \Gamma_{f_{t_0}} = \{0\}$  where  $T_a \Gamma_{f_{t_0}}$  denotes the tangent space at  $a$ , and so  $\deg(M \cdot \Gamma_{f_{t_0}}) = 1$ . But since in the holomorphic case, the local uniform convergence is a convergence with

the tangents, we easily conclude that for sufficiently large indices  $\nu$ ,  $M$  is transversal to the manifold (near  $a$ )  $\Gamma_{f_t}$  and so  $\deg(M \cdot \Gamma_{f_t}) = 1$ , too (there are no multiplicities attached to the graphs). To be somewhat more precise, if  $a = (a', f_{t_0}(a'))$ , then

$$T_{(a', f_{t_\nu}(a'))} \Gamma_{f_{t_\nu}} \xrightarrow{K} T_{(a', f_{t_0}(a'))} \Gamma_{f_{t_0}}$$

and we apply [TW1] to conclude that  $M$  intersects  $\Gamma_{f_t}$  transversally.  $\square$

Recall (cf. [D1]–[D3]) that if  $f \in \mathcal{O}_c(A)$  does not vanish identically on any irreducible component of  $A$ , where  $A$  is a pure  $k$ -dimensional analytic subset of a domain  $D \subset \mathbb{C}^m$ , then we define the *cycle of zeroes* as the Draper proper intersection cycle ([Dr])

$$Z_f := \Gamma_f \cdot (D \times \{0\}).$$

In the same way we may define the *fibre cycle*, namely

$$[f^{-1}(f(a))] := \Gamma_f \cdot (D \times \{f(a)\})$$

and consider this as a cycle in  $D$ .

Now we can state the following Hurwitz-type theorem:

**Theorem 2.7.** *Let  $T$  be a connected topological space,  $A$  a pure  $k$ -dimensional analytic subset of some domain  $D \subset \mathbb{C}^m$ ,  $f: T \times A \rightarrow \mathbb{C}$  a continuous function such that for each  $t \in T$ ,  $f_t(x) := f(x, t)$  is  $c$ -holomorphic on  $A$ . Then if  $f_{t_0} \not\equiv 0$  on any irreducible component of  $A$  and  $f_{t_0}^{-1}(0) \neq \emptyset$ , we have*

$$Z_{f_t} \xrightarrow{T} Z_{f_{t_0}}, \quad t \rightarrow t_0.$$

*Proof.* By the previous Proposition we have

$$\Gamma_{f_t} \xrightarrow{T} \Gamma_{f_{t_0}}.$$

Of course,  $f_{t_0}^{-1}(0)$  is a hypersurface (cf. the identity principle from [D2]) which means that the intersection  $\Gamma_{f_{t_0}} \cap (D \times \{0\})$  is proper (i.e. of the minimal dimension possible:  $k - 1$ ). By [T] Lemma 3.5 we conclude that for any sequence  $t_\nu \rightarrow t_0$ ,

$$\Gamma_{f_{t_\nu}} \cdot (D \times \{0\}) \xrightarrow{T} \Gamma_{f_{t_0}} \cdot (D \times \{0\}).$$

This ends the proof.  $\square$

**Corollary 2.8.** *Let  $g \in \mathcal{O}_c(A)$ ,  $g \neq \text{const.}$  on any irreducible component of  $A \subset D$ , where  $A$  is pure  $k$ -dimensional. Then for any  $t_0 \in A$ ,*

$$[g^{-1}(t)] \xrightarrow{T} [g^{-1}(t_0)], \quad t \rightarrow t_0.$$

*Proof.* Let  $f: A \times \mathbb{C} \ni (x, t) \mapsto g(x) - t \in \mathbb{C}$ . By [D2], we conclude that all the nonempty fibres of  $g$  have pure dimension  $k - 1$ . Then  $f$  satisfies the

assumptions of the preceding Theorem and

$$\begin{aligned} Z_{f_t} &= \Gamma_{f_t} \cdot (D \times \{0\}) = \\ &= \Gamma_g \cdot (D \times \{t\}) = \\ &= [g^{-1}(t)], \end{aligned}$$

since  $\Phi(x, s) = (x, s + t)$  is an automorphism of  $D \times \mathbb{C}$  sending  $\Gamma_{f_t}$  to  $\Gamma_g$  and  $D \times \{0\}$  to  $D \times \{t\}$ . This ends the proof.  $\square$

Before the next corollary recall that for any positive cycle  $Z = \sum \alpha_i S_i$  we define its *local degree* at  $a \in |Z|$  as  $\deg_a Z := \sum \alpha_i \deg_a S_i$ , where  $\deg_a S_i$  is the usual local degree (Lelong number) with the convention that  $\deg_a S_i = 0$  if  $a \notin S_i$ .

**Corollary 2.9.** *Under the assumptions of the preceding Theorem suppose in addition that  $f_t(a) = 0$  for all  $t \in T$  and some fixed  $a \in A$ . Then for all  $t$  close enough to  $t_0$ ,*

$$\deg_a Z_{f_t} \leq \deg_a Z_{f_{t_0}},$$

for the local degrees at  $a$ .

*Proof.* Take any affine subspace  $L$  through  $a$ , of dimension  $m - k + 1$  and such that

$$L \cdot Z_{f_{t_0}} = \deg_a Z_{f_{t_0}} \cdot \{a\}.$$

Then by Theorem 2.7 together with [T] Lemma 3.5,

$$L \cdot Z_{f_t} \xrightarrow{T} L \cdot Z_{f_{t_0}}$$

which ends the proof, since

$$L \cdot Z_{f_t} = \sum_{b \in L \cap f_t^{-1}(0)} i(L \cdot Z_{f_t}, b) \{b\}$$

and for each Draper intersection index (multiplicity)  $i(L \cdot Z_{f_t}, b)$  we have

$$i(L \cdot Z_{f_t}, b) \geq \deg_b Z_{f_t},$$

for  $\deg_b L = 1$ . Therefore, we obtain by the convergence, for all  $t$  sufficiently close to  $t_0$ ,

$$\begin{aligned} \deg_a Z_{f_{t_0}} &= \deg(L \cdot Z_{f_{t_0}}) = \\ &= \deg(L \cdot Z_{f_t}) = \\ &= \sum_{b \in L \cap f_t^{-1}(0)} i(L \cdot Z_{f_t}, b) \{b\} \geq \\ &\geq i(L \cdot Z_{f_t}, a) \{a\} \geq \deg_a Z_{f_t}, \end{aligned}$$

as  $a \in L \cap f_t^{-1}(0)$  (for all  $t$ ).  $\square$

### 3. ON THE ŁOJASIEWICZ INEQUALITY AND THE TOTAL DEGREE

We recall one result from [D3] which is the basis which we shall work upon.

**Theorem 3.1** ([D3] Theorem 2.3). *Let  $f: \Omega \rightarrow \mathbb{C}$  be holomorphic in a (connected) neighbourhood  $\Omega$  of  $0 \in \mathbb{C}^m$ . If  $f$  is non-constant and  $f(0) = 0$  then there is a neighbourhood  $U$  of zero such that the following Łojasiewicz inequality holds:*

$$|f(x)| \geq \text{const} \cdot \text{dist}(x, f^{-1}(0))^{\text{ord}_0 f}, \quad x \in U$$

where  $\text{ord}_0 f$  denotes the order of vanishing of  $f$  at zero. Moreover, this is the best exponent possible.

As before we consider the intersection cycle of zeroes  $Z_f = \Gamma_f \cdot (\Omega \times \{0\})$ .

**Proposition 3.2** ([D3] Proposition 2.1). *In the setting introduced above,  $\deg_0 Z_f = \text{ord}_0 f$ .*

We easily generalize these results to c-holomorphic functions, although only in a weak sense (compare the following theorem with the results of [D4]). Consider a pure  $k$ -dimensional ( $k \geq 2$ ) analytic subset  $A$  of a neighbourhood  $\Omega$  of  $0 \in \mathbb{C}^m$  with  $0 \in A$ . Assume that  $f \in \mathcal{O}_c(A)$  satisfies  $f(0) = 0$  and does not vanish identically on any irreducible component of  $A$  containing zero.

**Theorem 3.3.** *In the c-holomorphic setting introduced above, there is a neighbourhood  $W$  of zero such that*

$$|f(z)| \geq \text{const} \cdot \text{dist}(z, f^{-1}(0))^{\deg_0 Z_f \cdot \deg_0 f^{-1}(0)}, \quad z \in W \cap A.$$

*Proof.* Write  $\mathbb{C}^m = \mathbb{C}^{k-1} \times \mathbb{C}^{m-k+1}$  with coordinates  $(x, y)$ .

We may assume that the coordinates are chosen in such a way that the projection  $\pi(x, y) = x$  onto the first  $k-1$  coordinates is proper on  $Z := f^{-1}(0) \cap (U \times V)$  with covering number equal to the local degree  $\deg_0 f^{-1}(0) =: d$ . Here  $U \times V$  is a neighbourhood of the origin satisfying  $(\{0\} \times \bar{V}) \cap f^{-1}(0) = \{0\}$ .

Applying Proposition 2.2 from [CgT] we find a holomorphic mapping  $F: U \times \mathbb{C}^{m-k+1} \rightarrow \mathbb{C}^p$  such that  $F^{-1}(0) = f^{-1}(0) \cap (U \times V)$  and

$$(*) \quad \|F(x, y)\| \geq \text{dist}((x, y), Z)^d, \quad (x, y) \in U \times \mathbb{C}^{m-k+1}.$$

If we write  $F = (F_1, \dots, F_p)$  we observe that  $F_j^{-1}(0) \cap A \supset f^{-1}(0) \cap (U \times V)$  for all  $j$ . The intersection of the graph  $\Gamma_f$  with  $\Omega \times \{0\}$  being proper, we can now apply the c-holomorphic Nullstellensatz from [D3]. In other words, we find a neighbourhood  $W \subset U \times V$  of zero and  $p$  c-holomorphic functions  $h_j$  on  $W \cap A$  for which

$$(**) \quad F_j^\delta = h_j f \text{ on } A \cap W, \quad j = 1, \dots, p$$

with  $\delta = \deg_0 Z_f$ .

Combining  $(*)$  and  $(**)$  we eventually obtain the inequality looked for.  $\square$



**Proposition 3.4.** *Under the assumptions of the previous theorem,*

$$\deg_0 Z_f \cdot \deg_0 f^{-1}(0) \geq \text{ord}_0 f.$$

*Proof.* This follows from Lemma 4.8 in [D1].  $\square$

Using Corollary 2 and Proposition 3.2 we easily obtain

**Lemma 3.5.** *If  $f = f(t, x) \in \mathcal{O}_{k+m}$  is such that  $f_t(0) := f(t, 0) = 0$  for all  $t$  small enough and  $f_0 = f(0, \cdot)$  is non-constant, then*

$$\text{ord}_0 f_t \leq \text{ord}_0 f_0$$

*for all  $t$  sufficiently close to zero.*

**Example 3.6.** The inequality may be strict as we easily see by taking  $f(t, x) = tx + x^2$ ; then for  $t \neq 0$ ,  $\text{ord}_0 f_t = 1 < \text{ord}_0 f_0 = 2 = \text{ord}_0 f$ . But of course there is no direct relation with  $\text{ord}_0 f$ , it suffices to take  $f(t, x) = tx + x^3$  in order to have  $\text{ord}_0 f_t = 1 < \text{ord}_0 f = 2 < \text{ord}_0 f$ .

The proof of Theorem 3.1 suggests the following result.

**Proposition 3.7.** *Let  $V \times W \subset \subset \mathbb{C}^{m-1} \times \mathbb{C}$  be a bounded, connected neighbourhood of zero (a polydisc) and let  $P \in \mathcal{O}(V)[t]$  be unitary and such that  $P^{-1}(0) \subset (V \times W)$  projects properly onto  $V$ . Then in  $V \times W$  there is*

$$|P(x, t)| \geq \text{dist}((x, t), P^{-1}(0))^\delta$$

*with  $\delta = \deg((\{0\}^{m-1} \times W) \cdot Z_P)$ .*

*Proof.* Recall from [D3] that  $Z_P = \sum \alpha_j S_j$  where  $S_j$  are the irreducible components of  $P^{-1}(0)$  and  $\alpha_j = \min\{\text{ord}_z P \mid z \in \text{Reg} S_j\}$  is the generic order of vanishing of  $P$  along  $S_j$ . Note that each  $S_j$  projects onto the whole of  $V$ .

Now, since the intersections  $(\{x\} \times W) \cap P^{-1}(0)$  are proper, by [T] (see also [Ch]) we conclude that for any  $x_\nu \rightarrow 0$  we have

$$(\{x_\nu\}^{m-1} \times W) \cdot Z_P \xrightarrow{T} (\{0\}^{m-1} \times W) \cdot Z_P$$

and so  $\deg((\{0\}^{m-1} \times W) \cdot Z_P) = \delta$  for sufficiently large  $\nu$ .

Observe that for the generic  $x \in V$  we have the following situation:  $\{x\} \times W$  intersects  $P^{-1}(0)$  transversally at  $d$  regular points  $b^{(i)} = (x, t^{(i)})$ , where  $d$  is the multiplicity of the branched covering  $P^{-1}(0) \rightarrow V$ , each of these points belongs to exactly one  $S_j$ , all the  $S_j$ 's appear in this assignment, and  $\text{ord}_{b^{(i)}} P = \alpha_j$  for the unique  $j$  such that  $b^{(i)} \in S_j$ . Therefore, we may write

$$\delta = \sum_{b \in (\{x\} \times W) \cap P^{-1}(0)} \text{ord}_b P.$$

On the other hand, for any such point  $x$  we have

$$P(x, t) = \prod_{i=1}^d (t - t^{(i)})^{n_i}$$

with  $n_i$  independent of the point chosen. We observe that  $n_i = \text{ord}_{b^{(i)}} P$ . Indeed, if we write  $\{x\} \times W$  as the zero-set of an affine mapping  $\ell = (\ell_1, \dots, \ell_{m-1})$  restricted to  $V \times W$ , then the transversality of the intersection  $(\{x\} \times W) \cap P^{-1}(0)$  implies by the Tsikh-Yuzhakov result (see [Ch]) that the multiplicity  $m_{b^{(i)}}(P, \ell)$  at each point  $b^{(i)}$  of the proper mapping germ  $(P, \ell)$  is equal to the product of the orders of  $P$  and the  $\ell_j$ 's, i.e. to  $\text{ord}_{b^{(i)}} P$ . On the other hand, by [Ch] p. 107-108 we easily see that

$$m_{b^{(i)}}(P, \ell) = \text{ord}_{t^{(i)}} P|_{\{x\} \times W} = n_i.$$

Therefore,  $\delta = \sum_{i=1}^d n_i$ . This allows us to write, for the generic  $x \in V$ , the following inequalities:

$$\begin{aligned} |P(x, t)| &= \prod_{i=1}^d |t - t^{(i)}|^{n_i} = \\ &= \prod_{i=1}^d \|(x, t) - (x, t^{(i)})\|^{n_i} \geq \\ &\geq \text{dist}((x, t), P^{-1}(0))^{\sum_{i=1}^d n_i}. \end{aligned}$$

Extending this by continuity to the whole of  $V \times W$  ends the proof.  $\square$

*Remark 3.8.* The proof above is in fact an extrapolation of the proof of Theorem 3.1, where we use the Weierstrass Preparation in a neighbourhood of zero such that  $(\{0\} \times W) \cap f^{-1}(0) = \{0\}$  and  $\text{ord}_0 f = \text{ord}_0 P$ .

**Corollary 3.9.** *If  $f: V \times W \rightarrow \mathbb{C}$  is a holomorphic function such that  $f^{-1}(0)$  projects properly onto  $V$ , then for some possibly smaller neighbourhood  $U \subset V \times W$  of zero,  $f$  satisfies the Łojasiewicz inequality in  $U$  with exponent  $\deg((\{0\} \times W) \cdot Z_f)$ .*

*Proof.* In  $V \times W$  we can apply the Weierstrass Preparation Theorem and write  $f = hP$  with a holomorphic function  $h$  such that  $h^{-1}(0) = \emptyset$ . Shrinking the neighbourhood (actually, we need only to shrink  $V$  if any), we may assume that  $\inf |h| > 0$ . Then  $Z_f = Z_P$ , since  $\text{ord}_b f = \text{ord}_b P$ . The preceding Proposition gives the result.  $\square$

#### 4. THE ŁOJASIEWICZ INEQUALITY WITH PARAMETER

Eventually, we are ready to prove the main result.

**Theorem 4.1.** *Assume that  $f: T \times \Omega \rightarrow \mathbb{C}$  is a continuous function where  $T$  is a locally compact, connected topological space,  $\Omega \subset \mathbb{C}^m$  is a domain, and for all  $t \in T$ ,  $f_t \in \mathcal{O}(\Omega)$  does not vanish identically. Assume moreover that  $0 \in \Omega$  and  $f_t(0) = 0$  for any  $t$ . Then there is a neighbourhood  $U \subset \Omega$  of zero such that, for all  $t$  close enough to  $t_0$ ,*

$$|f_t(x)| \geq c(t) \text{dist}(x, f_t^{-1}(0))^\alpha, \quad x \in U$$

where  $c(t) > 0$  is a constant depending on the parameter, but the exponent

$$\alpha = \text{ord}_0 f_{t_0}$$

is uniform.

*Proof.* By Theorem 2.7 we know in particular that  $f_t^{-1}(0) \xrightarrow{K} f_{t_0}^{-1}(0)$ . Of course these sets are hypersurfaces. The type of convergence implies that we can choose coordinates in  $\mathbb{C}^m$  in such a way that for some neighbourhood  $V \times W \subset \mathbb{C}^{m-1} \times \mathbb{C}$  of zero,  $V$  connected and  $W$  a disc, we have

$$f_t^{-1}(0) \cap (V \times \partial W) = \emptyset$$

for all  $t$  close enough to  $t_0$ . This means that the zero-sets intersected with  $V \times W$  project properly onto  $V$ . Moreover, we may assume that

$$(\{0\}^{m-1} \times W) \cdot Z_{f_{t_0}} = \text{ord}_0 f_{t_0} \{0\}.$$

In the situation considered, the proof of Proposition 3.7 shows that the Łojasiewicz inequality for  $f_{t_0}$  is satisfied in  $V \times W$  with the exponent  $d_t = \deg((\{0\} \times W) \cdot Z_{f_t})$ :

$$(*) \quad |f_t(x)| \geq c(t) \text{dist}(x, f_t^{-1}(0))^{d_t}, \quad x \in V \times W$$

where  $c(t) > 0$  is a constant.

But then, for  $t$  close enough to  $t_0$ , the numbers  $d_t$  fortunately coincide with  $(\{0\} \times W) \cdot Z_{f_{t_0}} = \text{ord}_0 f_{t_0}$  by the convergence (Theorem 2.7).

This ends the proof.  $\square$

It seems hard to obtain a satisfactory c-holomorphic counter-part to this Theorem due to the use of the Nullstellensatz with parameter. The best we were able to obtain is the following Theorem.

**Theorem 4.2.** *Assume that  $f: T \times A \rightarrow \mathbb{C}$  is a continuous function where  $T$  is a locally compact, connected topological space,  $A$  is a pure  $k$ -dimensional analytic subset of an open set  $\Omega \subset \mathbb{C}^m$ ,  $0 \in A$ , and for all  $t \in T$ ,  $f_t \in \mathcal{O}_c(A)$  does not vanish identically on any irreducible component of  $A$  through zero. Assume moreover that  $f_t(0) = 0$  for any  $t$ . Then there is a neighbourhood  $U \subset \Omega$  of zero such that, for all  $t$  close enough to  $t_0$ ,*

$$|f_t(x)| \geq c(t) \text{dist}(x, f_t^{-1}(0))^\alpha, \quad x \in A \cap U$$

where  $c(t) > 0$  is a constant depending on the parameter, but the exponent

$$\alpha = (\deg_0 Z_{f_{t_0}})^2$$

is uniform.

*Proof.* We give the proof in several steps.

**Step 1.** Choose coordinates in  $\mathbb{C}^m$  in such a way that  $A$  projects properly onto the first  $k$  coordinates and, moreover,

$$i((\{0\}^{k-1} \times \mathbb{C}^{m-k+1}) \cdot Z_{f_{t_0}}; 0) = \deg_0 Z_{f_{t_0}}.$$

Let  $\ell: \mathbb{C}^m \rightarrow \mathbb{C}^{k-1}$  be the linear epimorphism whose kernel is exactly  $\{0\}^{k-1} \times \mathbb{C}^{m-k+1}$ . Write

$$\varphi_t: A \ni x \mapsto (f_t(x), \ell(x)) \in \mathbb{C} \times \mathbb{C}^{k-1}$$

for  $t \in T$ . Fix a polydisc  $V \times W \subset \mathbb{C}^{k-1} \times \mathbb{C}^{m+k-1}$  centred at zero such that

$$(\{0\}^{k-1} \times \overline{W}) \cap f_{t_0}^{-1}(0) = \{0\}.$$

In particular we may assume that  $f_{t_0}^{-1}(0)$  projects properly onto  $V$ .

**Step 2.** The latter intersection corresponds to  $(\overline{V} \times \overline{W} \times \{0\}^k) \cap \Gamma_{\varphi_{t_0}}$  which means that there is a polydisc  $P \subset \mathbb{C}^k$  such that the pure  $k$ -dimensional analytic set  $(V \times W \times P) \cap \Gamma_{\varphi_{t_0}}$  projects properly *onto*  $P$  along  $V \times W$ . In other words,  $\varphi_{t_0}|_{(V \times W) \cap A}$  is proper with image  $P$ .

As in Lemma 2.5, the continuity of

$$\Phi: T \times A \ni (t, x) \mapsto \varphi_t(x) \in \mathbb{C}^k$$

implies the Kuratowski convergence of the graphs  $\Gamma_{\varphi_t} \xrightarrow{K} \Gamma_{\varphi_{t_0}}$  as  $t \rightarrow t_0$ . Therefore, by the same argument as in Proposition 1.7, we conclude that for all  $t$  close enough to  $t_0$ , the restrictions of the natural projection

$$\pi_t: (V \times W \times P) \cap \Gamma_{\varphi_t} \rightarrow P$$

are branched coverings. In particular, all these  $\varphi_t$  have the same image  $P$ . Let  $q_t$  denote the multiplicity of the branched covering  $\varphi_t|_{A \cap (V \times W)}$ .

**Step 3.** By the choice of  $V \times W$  and Theorem 2.7, we know (cf. the proof of the previous Theorem) that for all  $t$  close enough to  $t_0$ , the zero-sets  $f_t^{-1}(0) \cap (V \times W)$  project properly onto  $V$ . Let  $d_t$  denote the multiplicity of such a branched covering.

Since by Theorem 2.7 we know that the cycles of zeros of the restrictions  $f_t|_{A \cap (V \times W)}$  converge with  $t \rightarrow t_0$  in the sense of Tworzewski, we easily conclude from [T] Lemma 3.5 and [TW1] that

$$(\star) \quad d_{t_0} \leq d_t \leq \deg((\{0\} \times W) \cdot Z_{f_t}) = \deg((\{0\} \times W) \cdot Z_{f_{t_0}}) = \deg_0 Z_{f_{t_0}}.$$

On the other hand, we observe that  $q_t = \deg((\{0\} \times W) \cdot Z_{f_t})$  and so

$$(\star\star) \quad q_t \leq \deg_0 Z_{f_{t_0}}.$$

Indeed, it is easy to see that  $q_t$  is in fact the multiplicity of the projection

$$\pi: \mathbb{C}^{k-1} \times \mathbb{C}^{m-k+1} \times \mathbb{C} \ni (u, v, w) \mapsto (w, u) \in \mathbb{C} \times \mathbb{C}^{k-1}$$

over  $P$  when restricted to  $\Gamma_t := \Gamma_{f_t} \cap (V \times W \times \mathbb{C})$ . This, in turn, by the classical Stoll Formula, is the total degree of the intersection cycle  $\pi^{-1}(0) \cdot \Gamma_t$ . In other words,

$$q_t = \deg((V \times W \times \{0\}) \cdot \Gamma_t).$$

However, in view of [TW2] Theorem 2.2, we can write

$$\begin{aligned} (V \times W \times \{0\}) \cdot \Gamma_t &= (\{0\} \times W) \cdot_{V \times W \times \{0\}} ((V \times W \times \{0\}) \cdot \Gamma_t) = \\ &= (\{0\} \times W) \cdot Z_{f_t|_{A \cap (V \times W)}} = \\ &= (\{0\} \times W) \cdot Z_{f_t}. \end{aligned}$$

**Step 4.** As in the proof of Theorem 3.3, by [CgT] Proposition 2.2 we know that for each  $t$  close to  $t_0$  there are  $p_t = d_t(m - k) + 1$  holomorphic

functions  $F_{t,j}: V \times \mathbb{C}^{m-k+1} \rightarrow \mathbb{C}$  whose common zeroes form coincide with the set  $f_t^{-1}(0) \cap (V \times W)$  and for which

$$\|(F_{t,1}, \dots, F_{t,p_t})(x)\| \geq \text{dist}(x, f_t^{-1}(0) \cap (V \times W))^{d_t}$$

for all  $x \in V \times W$ .

Now, we can apply Lemma 3.1 from [D3] (compare [PT]) in order to get *on the whole* of  $A \cap (V \times W)$ ,

$$F_{t,j}^{q_t} = h_{t,j} f_t, \quad j = 1, \dots, p_t,$$

with some functions  $h_{t,j} \in \mathcal{O}_c(A \cap (V \times W))$ .

This leads to the inequalities

$$(\#) \quad |f_t(x)| \geq c(t) \text{dist}(x, f_t^{-1}(0))^{p_t q_t}, \quad x \in A \cap (V \times W)$$

for all  $t$  close to  $t_0$  and some constants  $c(t) > 0$ .

**Step 5.** Thanks to the continuity of the zero-sets (cf. Theorem 2.7), Proposition 2.3 (cf. Remark 2.4) allows us to choose an arbitrarily small neighbourhood  $T_0$  of  $t_0$  and a neighbourhood  $U \subset V \times W$  of zero such that for all  $t \in T_0$  and all  $x \in U$ , we have

$$\text{dist}(x, f_t^{-1}(0)) < 1.$$

Therefore, we may increase *ad libitum* the exponent in  $(\#)$ , provided  $x \in A \cap U$ . The estimates  $(\star)$  and  $(\star\star)$  end the proof.  $\square$

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